Shape Heuristics in Aristotelian Diagrams

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Abstract. Aristotelian diagrams have a long and rich history in philosophical logic. Today, they are widely used in nearly all disciplines dealing with logical reasoning. Logical geometry is concerned with the theoretical study of these diagrams, from both a logical and a visual/geometrical perspective. In this paper, we argue that the concrete shape of Aristotelian diagrams can be of great heuristic value in logical geometry. A diagram's shape can be used to visually represent certain logical properties and relations, and hence, through its shape, the diagram can help us to better understand these properties and relations and reason about them. These claims are explained and illustrated by means of a series of examples, consisting mainly of (i) Aristotelian diagrams that represent an entire Boolean algebra of formulas, and (ii) complementarities between Aristotelian diagrams.

Keywords. shape heuristics, Aristotelian diagram, logical geometry, square of opposition, Jacoby-Sesmat-Blanché hexagon, rhombic dodecahedron

1. Introduction

Aristotelian diagrams are compact visual representations of the elements of some logical or conceptual field, and the logical relations holding between them. These diagrams have a long and rich history in philosophical logic [1,2,3,4]. Today, they are still widely used in logic [5,6,7], but also in fields such as cognitive science, linguistics, philosophy, neuroscience, law and computer science [8,9,10,11,12,13] (see [14, Section 1] for more examples). In sum, then, it seems fair to conclude that Aristotelian diagrams have come to serve "as a kind of *lingua franca*" [15, p. 81] for a highly interdisciplinary community of researchers who are all concerned, in some way or another, with logical reasoning.

Logical geometry[3] systematically investigates Aristotelian diagrams as objects of independent interest (regardless of their role as lingua franca), from both a logical and a visual/geometric perspective. Typical logical topics include the information contents and logic-dependence of Aristotelian diagrams [14,16]. Typical visual/geometrical features of Aristotelian diagrams include length, dimensionality, perpendicularity, collinearity, convexity—or more generally: the diagram's *shape*. For example, given a fragment of four logical formulas and the Aristotelian relations holding between them, this can be visualized by means of a square (as is usually done), or alternatively a rectangle or a rhom-

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bus, or even—moving from two-dimensional to three-dimensional space—by means of a tetrahedron (or some irregular variant thereof).

The main aim of this paper is to argue that shape plays an important heuristic role in logical geometry. Building on earlier work [17,18,19], we will show that the shape of an Aristotelian diagram can be used to visually represent certain logical properties and relations, and consequently, through its shape, the diagram can help us to better understand these properties and relations, and reason about them. We will further substantiate and illustrate these claims, by presenting a series of examples and discussing the precise heuristic role that shape plays in each of them.

The paper is organized as follows. In Section 2, we first introduce Aristotelian diagrams and their key properties, and then argue that a diagram's shape can be of great heuristic value in logical geometry. In Section 3 we discuss the heuristic role of shape in Aristotelian diagrams that represent an entire Boolean algebra, and in Section 4 we discuss various shape heuristics for complementarities between Aristotelian diagrams. Finally, Section 5 wraps things up, and mentions some questions for further research.

2. Aristotelian Diagrams and the Heuristic Value of Shape

An Aristotelian diagram visualizes a set of logical formulas and the Aristotelian relations between them (see Fig. 1 for a basic example). These relations are defined as follows (relative to some given logical system S, which is taken to have the usual Boolean connectives, and a model-theoretic semantics $\models$): the formulas φ and ψ are said to be

S-*contradictory*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
S-*contrary*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$,
S-*subcontrary*	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
in S-*subalternation*	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$.

This is a straightforward formalization of the traditional perspective on the Aristotelian relations, according to which two formulas are, for example, contrary iff they cannot be true together (cf. $\mathsf{S} \models \neg(\varphi \wedge \psi)$), but can be false together (cf. $\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$).

Furthermore, almost all Aristotelian diagrams that have appeared in the literature impose the following additional constraints on the formulas that are visualized: these formulas are supposed to be (i) contingent and (ii) pairwise non-equivalent, and (iii) they come in contradictory pairs (i.e. for a given formula φ, the diagram contains both φ and $\neg\varphi$). The historical and technical motivations for imposing these constraints are discussed in more detail in [16, Subsection 2.1]. For our current purposes, it suffices to note that although the contingency constraint means that an Aristotelian diagram should not contain the non-contingent formulas $\bot$ and $\top$ at all, some authors [20,21] prefer to think of them as coinciding in the center of the diagram. From this perspective, $\bot$ and $\top$ still do not occupy any real vertices of the diagram, but they are 'hidden' in its center (which is itself *not* a separate vertex of the diagram). For example, in the Aristotelian square in Fig. 1(b), $\bot$ and $\top$ can be thought of as coinciding in the square's center, where the two diagonals intersect each other.

An Aristotelian diagram trivially depends on the formulas that it visualizes: Aristotelian diagrams that visualize distinct (sets of) formulas are themselves distinct. Furthermore, it has recently been realized that the logical system also plays a crucial role

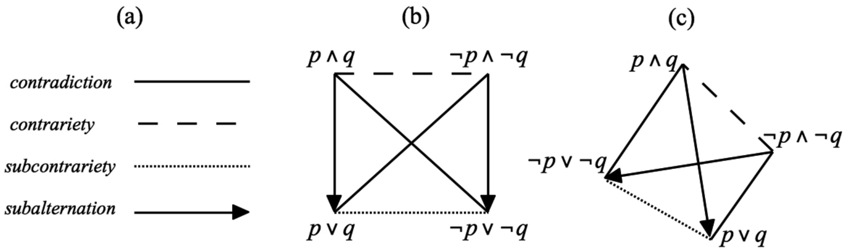

Figure 1. (a) Code for visualizing the Aristotelian relations, (b) an Aristotelian square with CPL-formulas, and (c) an alternative (three-dimensional) visualization by means of a tetrahedron.

[14,22], since the Aristotelian relations are defined with respect to a such a system.[4] After these two parameters (formulas and logical system) have been fixed, the *logical* properties of the Aristotelian diagram are fully determined. From a *visual/geometrical* perspective, however, the Aristotelian diagram is still seriously underspecified—i.e. there are still several 'design choices' that need to be made when constructing the actual diagram.[5]

Consider, for example, the propositional formulas $p \wedge q$, $p \vee q$, $\neg p \wedge \neg q$ and $\neg p \vee \neg q$, and suppose that we are working in the system of classical propositional logic (CPL). The Aristotelian relations holding between these formulas are fully determined (e.g. $p \wedge q$ and $\neg p \wedge \neg q$ are CPL-contrary, etc.), and we are only left with the task of visualizing these formulas and relations in an appropriate manner. This, however, can be done in various ways: the most common approach is to make use of a square, as in Fig. 1(b), but we could also make use of rectangle, a rhombus, or even move from two-dimensional to three-dimensional shapes, and make use of a tetrahedron, as in Fig. 1(c). These various diagrams contain the same (logical) information, and hence they can be said to be *informationally equivalent* to each other [26]. One can also ask, however, whether these distinctly shaped diagrams are also *cognitively* or *computationally equivalent* to each other [26], i.e. whether the information that they contain can equally easily be extracted or inferred in all cases. Are all these diagrams equally helpful visualizations (of the given set of formulas in the given logical system), or can some of them be argued to be 'better' than the others (and on what grounds)?

In the next two sections, we will show that in many cases, the concrete shape of an Aristotelian diagram can have a powerful *heuristic* function. Roughly speaking, the idea is that the (sets of) formulas represented by Aristotelian diagrams have various interesting properties and relations amongst each other, and a diagram's shape can be used to visualize these properties and relations, thereby enabling the user to better understand them and reason about them. Following the work of Gurr [27,28,29] on isomorphisms in diagrammatic representations,[6] a well-designed Aristotelian diagram can be said to embody a kind of 'isomorphism' between the (set of) formulas' logical properties and the diagram's visual/geometrical properties. The diagram thus simultaneously engages the user's logical and visual cognitive systems, and thereby facilitates inferential or heuris-

[4]Consider, for example, the modal formulas $\Box p$ and $\Box\neg p$. In the modal system D, these formulas are contrary, but in the minimal normal modal system K, they do not stand in any Aristotelian relation at all [23].

[5]As described in [24,25], a similar situation arises in the study of Euler diagrams.

[6]The cognitive effectiveness of diagrams has also been explained using notions that are closely related to *isomorphism*, such as *congruity* [30] and *iconicity* [31].

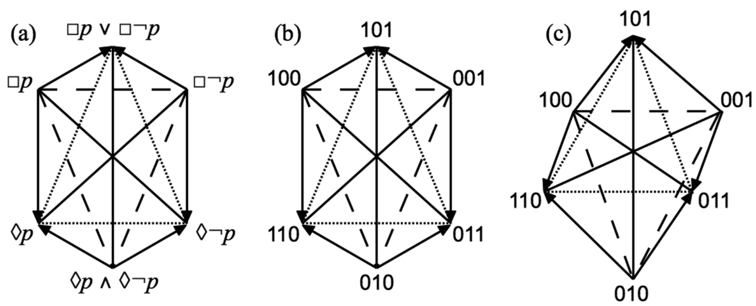

Figure 2. (a) JSB hexagon for a Boolean closed set of formulas in the modal system S5, (b) its representation using bitstrings of length 3, (c) an alternative (3D) visualization by means of an octahedron.

tic *free rides* [32]: since the logical properties are directly visually manifested in the diagram, it enables the user to grasp these properties with very little cognitive effort.

These considerations also imply a partial answer to the question concerning cognitive differences between informationally equivalent diagrams (such as those in Fig. 1(b–c)). If we have Aristotelian diagrams D_1 and D_2 (for a given set of formulas and logical system), and the shape of D_1 triggers more heuristics than that of D_2, then *ceteris paribus*[7] D_1 can be said to be a better or more effective visualization (of the given formulas in the given logical system) than D_2.

3. Aristotelian Diagrams for Boolean Algebras

Our first series of case studies concerns Aristotelian diagrams that are *Boolean closed*, i.e. that contain all contingent Boolean combinations of their formulas. Equivalently, these diagrams can be characterized as visualizing an entire finite Boolean algebra, except for its top and bottom elements.[8] It is well-known that finite Boolean algebras have 2^n elements (with $n \in \mathbb{N}$), and can be represented as the powerset of an n-element set, or equivalently, as the set $\{0,1\}^n$ of bitstrings of length n.[9]

The first interesting case arises when $n = 3$, which yields $2^3 = 8$ formulas/bitstrings. After excluding $\top$ and $\bot$, we are thus left with $8 - 2 = 6$ *contingent* formulas/bitstrings, which stand in various Aristotelian relations to each other, and constitute a diagram commonly known as a *(strong) Jacoby-Sesmat-Blanché* (JSB) diagram [16,17,18,22]. This diagram can be drawn using various geometric shapes. By far the most common one is a two-dimensional *hexagon*, as shown in Fig. 2(a–b). One of the alternatives, which can be found in [20,33], is to make use of a three-dimensional *octahedron*, as shown in Fig. 2(c). These hexagon- and octahedron-shaped diagrams represent exactly the same logical formulas (or rather, their bitstring representations) and the Aristotelian relations holding between them, and hence they are informationally equivalent to each other. However, as will be shown below, these diagrams are certainly not cognitively equivalent, since the

[7] We do not mean to imply that factors other than heuristic value do not play any role at all. We only want to argue for the *relevance* of heuristic value in diagram design, not for the *irrelevance* of any other factors.

[8] Recall the constraint that Aristotelian diagrams should only contain contingent formulas.

[9] A systematic technique for mapping any set of formulas onto bitstrings is described in [22].

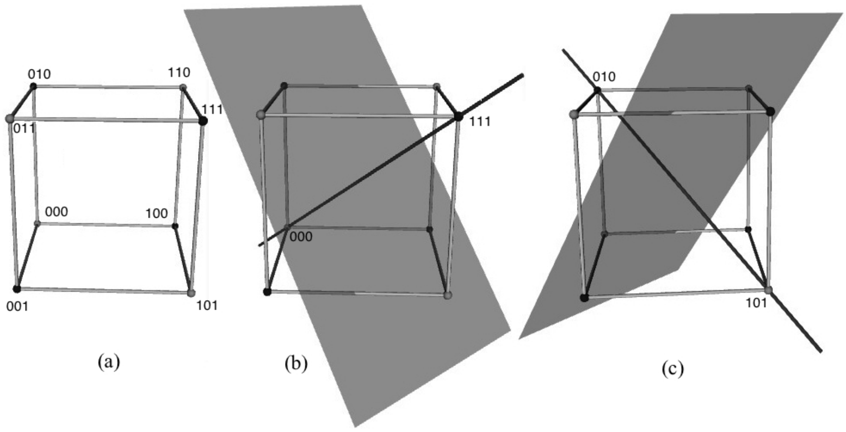

Figure 3. (a) 3-dimensional cube for the Boolean algebra of bitstrings of length 3, (b) vertex-first projection along the 111-000 axis, (c) vertex-first projection along the 101-010 axis.

hexagon enables more heuristic free rides—and is thus a more effective visualization—than the octahedron.

It is well-known that the Boolean algebra of bitstrings of length n can be represented as an n-dimensional hypercube: every bit position corresponds to a dimension, and the two values that it can take (0 and 1) define line segments in that dimension, which are edges of the hypercube. The bitstrings thus not only serve as Boolean/logical entities, but also as coordinates of the hypercube's vertices. In the case $n = 3$, we thus find a 3-dimensional hypercube, i.e. an 'ordinary' *cube*, which is shown in Fig. 3(a). We now consider the vertex-first projection of this cube along the axis defined by the non-contingent bitstrings 111 and 000, as shown in Fig. 3(b). The result of this projection is a hexagon, with the bitstrings occupying the relative positions as in the hexagon in Fig. 2(b) [17]. In other words, if it is visualized by means of a hexagon, then the JSB diagram for the Boolean algebra $\{0,1\}^3$ turns out to be the vertex-first projection of the well-known (hyper)cube representation of that same Boolean algebra.

Because of this geometric fact, the hexagon is a more effective diagram than the octahedron. First of all, the hexagonal shape serves to remind the user that she is working with (an isometric projection of) a Boolean cube, and thus with the full Boolean algebra $\{0,1\}^3$, i.e. with a Boolean closed set of formulas/bitstrings. Hence, there is an immediate connection between a visual property (being a hexagon) and a logical property (being Boolean closed). By contrast, the octahedral visualization of $\{0,1\}^3$ lacks this connection, since the octahedron is not the vertex-first projection of the cube.

One might object at this point that the octahedron is related to the cube in other geometrical ways; most importantly, the octahedron is *dual* to the cube. However, unlike taking a vertex-first projection, the process of dualizing the cube does not have the right geometric properties. For example, by dualizing the cube we go from a polyhedron with 8 vertices (the cube) to a polyhedron with 6 vertices (the octahedron), but the geometrical reason for this reduction from 8 to 6 vertices is the (convenient but ad hoc) fact that the cube has 6 faces (recall that dualization involves turning faces into vertices and vice versa). The octahedron's 6 vertices thus have very little to do with the cube's 8 original vertices. All this stands in sharp contrast to the hexagon. By taking the vertex-first projection of the cube, we also go from a polyhedron with 8 vertices (the cube) to a polygon with 6 vertices (the hexagon), but in this case, the geometrical reason for this reduction from 8 to 6 vertices is the highly relevant fact that two of the cube's vertices lie exactly on the projection axis, and hence will coincide with each other and lie in the center of the (two-dimensional) result of the projection (i.e. the hexagon). The hexagon's 6 vertices are thus closely related to the cube's 8 original vertices, viz. they correspond exactly to the cube's vertices that do *not* lie on the projection axis.

Furthermore, recall that the projection axis is defined by the non-contingent bitstrings 111 and 000. This justifies the intuitive idea (ascribed in Section 2 to [20,21]) that these non-contingent bitstrings are not entirely absent from an Aristotelian diagram, but rather coincide in its center of symmetry. Similarly, the 6 vertices of the cube that do *not* lie on the projection axis—and thus correspond to the 6 vertices of the hexagon—are exactly the contingent bitstrings in $\{0,1\}^3$; this reflects the constraint that Aristotelian diagrams should only contain contingent bitstrings. By contrast, the octahedral visualization cannot explain why the JSB diagram only contains contingent bitstrings, nor why the non-contingent bitstrings can be thought of as coinciding in the diagram's center of symmetry.

Finally, note that the Boolean algebra $\{0,1\}^3$ can not only be visualized by means of an Aristotelian diagram, but also by means of a *Hasse diagram*. In [17] it is shown that the hexagonal Hasse diagram (almost) coincides with the vertex-first projection of the cube for $\{0,1\}^3$ along the projection axis defined by the bitstrings 101 and 010 (or any other contradictory pair of contingent bitstrings); see Fig. 3(c). There is thus a very close relationship between the hexagonal Aristotelian and Hasse diagrams for $\{0,1\}^3$: both are vertex-first projections of one and the same cube, and their differences are entirely due to the different projection axes. By contrast, the octahedral visualization of the Aristotelian (or Hasse) diagram for $\{0,1\}^3$ does not allow for such a unified perspective.

To summarize: the hexagonal JSB diagram in Fig. 2(b) embodies a strong isomorphism between visual-geometric properties (e.g. being the vertex-first projection of a cube) and abstract-logical properties (e.g. being Boolean closed). This isomorphism has significant cognitive advantages, since it enables heuristic free rides: several logical properties, which might otherwise easily be forgotten or overlooked, are visually manifested in the diagram's shape, and thus can no longer escape the user's attention. By contrast, the octahedral JSB diagram in Fig. 2(c) does *not* embody an isomorphism, and is thus less powerful from a cognitive-heuristic perspective. Hence, even though these two diagrams are informationally equivalent, they are certainly not cognitively equivalent.

Similar remarks can be made when we move to bitstrings of length 4. There are $2^4 - 2 = 14$ contingent bitstrings of length 4, and there are various ways to visualize these bitstrings and the Aristotelian relations between them. The most commonly used diagram is the *rhombic dodecahedron* [5,20,34], or an irregular variant thereof [21,35].[10] An alternative visualization makes use of a *nested tetrahedron* [13]. Although these two diagrams are informationally equivalent, the former is the vertex-first projection of a hypercube [17], and is therefore cognitively more effective than the latter.

[10] See [18] for a comparison between the canonical rhombic dodecahedron and its irregular variants.

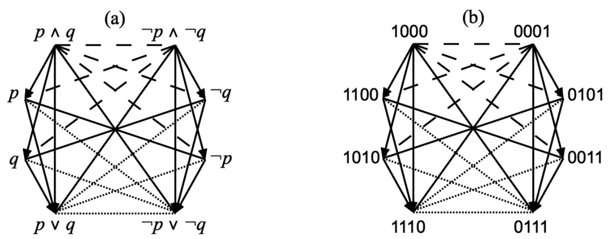

Figure 4. (a) Buridan octagon for propositional formulas, and (b) its representation by bitstrings of length 4.

4. Complementarities between Aristotelian Diagrams

We now turn to our second series of case studies, which concerns Aristotelian diagrams that are *not* Boolean closed, but can be represented by bitstrings of length 4. Logically speaking, this means that the Boolean closure of these diagrams is (isomorphic to) $\{0,1\}^4$, and hence, diagrammatically speaking, these diagrams can be seen as subdiagrams of the rhombic dodecahedron for $\{0,1\}^4$ (recall from Section 3 that the rhombic dodecahedron is the best visualization of $\{0,1\}^4$).

We begin by considering the Aristotelian diagram shown in Fig. 4(a), which shows 8 propositional formulas and the Aristotelian relations holding between them. This diagram is sometimes called a *Buridan diagram*, because the first Aristotelian diagram of this type can be found in the logical works of the 14th-century philosopher John Buridan (who used it in the context of syllogistics, rather than propositional logic) [3,4]. It is well-known that the Buridan diagram for propositional logic can be represented by bitstrings of length 4, as shown in Fig. 4(b) [22]. Buridan diagrams are usually visualized by means of an *octagon*, as in Fig. 4 [3,4,7,22]; an alternative visualization, which can be found in [18,19,34], makes use of a *rhombicube*, as in Fig. 5(a). These octagon- and rhombicube-shaped diagrams represent exactly the same logical formulas/bitstrings and the Aristotelian relations holding between them, and hence they are informationally equivalent to each other. However, just like in Section 3, we will show that they are certainly not cognitively equivalent, since one of them enables much more heuristic free rides than the other one.

First of all, note that in the rhombicube, the visual property of *verticality* perfectly represents the logical property of *level* (i.e. the number of 1-bits in the bitstring): the level-1 bitstrings (1000 and 0001) are at the top of the diagram, the level-3 bitstrings (1110 and 0111) are at the bottom of the diagram, and the level-2 bitstrings (1100, 1010, 0101 and 0011) all lie on a horizontal plane exactly in the middle. By contrast, in the octagon, this visual-logical isomorphism cannot be maintained: for example, in Fig. 4(b), 1010 is situated *lower* than 1100, even though these two bitstrings are of the same logical level. An obvious solution to this problem seems to involve putting all level-2 bitstrings on a horizontal line, exactly in between the horizontal line constituted by the level-1 bitstrings and the horizontal line constituted by the level-3 bitstrings. This is problematic, however, because the collinearity of the level-2 bitstrings entails that the Aristotelian relations between these bitstrings can no longer be clearly visualized; for example, the line representing the contradiction between 1010 and 0101 would overlap with the line representing the contradiction between 1100 and 0011 (see [17] for a more detailed discussion about the diagrammatic issues pertaining to verticality, level and collinearity).

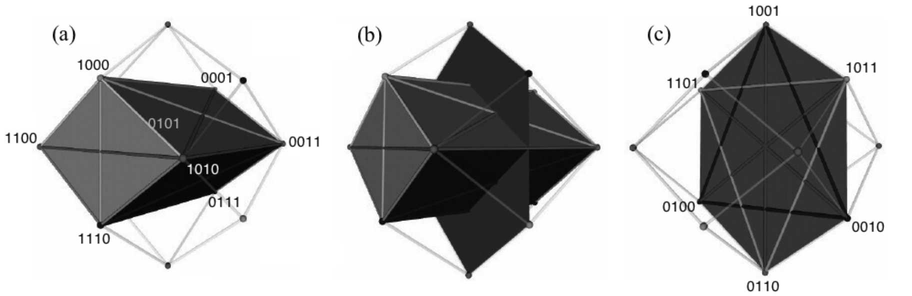

Figure 5. (a) Buridan rhombicube inside the rhombic dodecahedron, (b) geometric complementarity between Buridan rhombicube and JSB hexagon, (c) JSB hexagon inside the rhombic dodecahedron.

Secondly, as is clear from Fig. 5(a), the rhombicube is exactly the shape that a Buridan diagram takes when it is seen as a subdiagram inside the rhombic dodecahedron for $\{0,1\}^4$. In particular, the rhombic faces of the rhombicube coincide with the rhombic faces of the rhombic dodecahedron. Through its shape, the rhombicube visualization of the Buridan diagram thus immediately suggests that this diagram can be embedded inside the rhombic dodecahedron for $\{0,1\}^4$, and hence, that its formulas can be represented by bitstrings of length 4. By contrast, the octagon in Fig. 4(b) does not in any way suggest that it can be embedded inside the rhombic dodecahedron, and thus does not remind the user that its formulas can be represented by bitstrings of length 4.

Finally, it should be noted that the 8 bitstrings constituting the Buridan diagram all have different values in their first and fourth bit positions, which we will formalize by saying that they satisfy the constraint $pos_1 \neq pos_4$. Out of the 16 bitstrings in the Boolean algebra $\{0,1\}^4$, exactly 8 satisfy the constraint $pos_1 \neq pos_4$; these are the ones constituting the Buridan diagram. The remaining 6 bitstrings that do not satisfy this constraint—i.e., that satisfy the complementary constraint $pos_1 = pos_4$—constitute a JSB diagram.[11] There thus exists a *logical complementarity* between Buridan diagrams and JSB diagrams: the bitstrings constituting these two types of Aristotelian diagrams satisfy two complementary constraints on bit positions.

When a JSB diagram is embedded inside the rhombic dodecahedron, it has the shape of a hexagon, which cuts across that rhombic dodecahedron, as shown in Fig. 5(c).[12] Furthermore, this hexagon perfectly cuts across the Buridan rhombicube, as shown in Fig. 5(b); together, the Buridan rhombicube and the JSB hexagon form a perfect partition of the rhombic dodecahedron, i.e. there exists a *geometrical complementarity* between the Buridan rhombicube and the JSB hexagon. Hence, if the Buridan diagram is visualized by means of a rhombicube, then it is geometrically complementary to (the hexagonal visualization of) the JSB diagram, which explicitly reminds the user of the underlying logical complementarity between the two diagrams. By contrast, if the Buridan diagram is visualized by means of an octagon, then there is no geometrical complementarity with the JSB diagram, and the underlying logical complementarity runs the risk of being overlooked.

[11] There are actually 8, rather than 6, bitstrings that do not satisfy the constraint $pos_1 \neq pos_4$, but two of them are the non-contingent bitstrings 1111 (which has a 1 in its first and fourth bit positions) and 0000 (which has a 0 in both its first and fourth bit positions), which do not occur in Aristotelian diagrams (cf. Section 2).

[12] Note that this is yet another reason for visualizing a JSB diagram by means of a hexagon, rather than an octahedron, besides the reasons already given in Section 3.

To summarize: the rhombicube-shaped Buridan diagram in Fig. 5(a) embodies a strong isomorphism between geometric properties (e.g. verticality, complementarity to a hexagon) and logical properties (e.g. level, constraints on bitstrings). This isomorphism has significant cognitive advantages, since it enables heuristic free rides: several logical properties, which might otherwise easily be forgotten or overlooked, are visually manifested in the diagram's shape, and can thus no longer escape the user's attention. By contrast, the octagonal Buridan diagram in Fig. 4(b) does *not* embody an isomorphism, and is thus less powerful from a cognitive-heuristic perspective. Hence, even though these two diagrams are informationally equivalent, they are certainly not cognitively equivalent.

5. Conclusion

In recent years, logical geometry has studied a wide variety of Aristotelian diagrams. Most of these diagrams belong to fundamentally distinct families, but there are also pairs of diagrams that contain exactly the same logical formulas and relations, and are thus informationally equivalent to each other. We have argued that in such cases, the diagrams' *shape* often plays a crucial role in their cognitive-heuristic effectiveness, and can thus help us to determine whether the diagrams are also computationally equivalent to each other.

This claim has been substantiated and illustrated by two series of case studies. First of all, focusing on *Boolean closed* sets of formulas/bitstrings, we have argued extensively that the hexagon is a more effective visualization of $\{0,1\}^3$ than the octahedron, since it embodies a strong isomorphism between the bitstrings' abstract-logical properties and the diagram's visual-geometrical properties. Secondly, focusing on logical and geometrical *complementarities* between Aristotelian diagrams, we have argued that the Buridan diagram can more effectively be drawn as a rhombicube, rather than as an octagon.

In ongoing work [36], we are investigating informationally equivalent visualizations of other types of Aristotelian diagrams, such as the so-called Sherwood-Czeżowski diagram. The principles discussed in this paper (being Boolean closed, complementarities with other diagrams) do not suffice to make cognitively relevant distinctions between these visualizations, so other, more fine-grained criteria will have to be looked for.

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