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Algebraic and Cognitive Aspects of Presenting Aristotelian Diagrams

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Structure of the talk

1. Introduction
2. Preliminaries
3. Diagrams with 4 formulas
4. Diagrams with 6 formulas
5. Conclusion

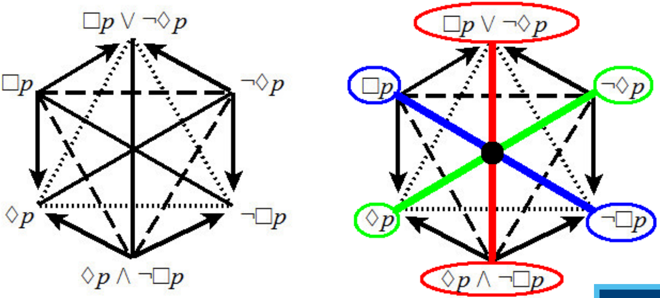
- recent years: renewed theoretical interest in the square of oppositions
  - extensions: e.g. hexagons, octagons, cubes, RDHs, etc.
  - families: e.g. Sesmat-Blanché hexagon vs. Sherwood-Czezowski hexagon
  - interrelations: e.g. 6 Sesmat-Blanché hexagons embedded inside RDH

$\Rightarrow$ strong identification between diagram and its formulas:
1 Aristotelian diagram $\leftrightsquigarrow$ 1 set of formulas

- different sets of formulas for 1 Aristotelian diagram
  - case studies on different decorations ($<$ logical systems, lexical fields)
  - e.g. Sesmat-Blanché hexagon for modal logic vs. subjective quantification

- different Aristotelian diagrams for 1 set of formulas ($\Rightarrow$ our focus today)
  - e.g. set of 4 formulas: square with subalternations $\downdownarrows$ vs. $\rightrightarrows$ vs. $\upuparrows$ vs. $\leftleftarrows$
  - e.g. set of 6 formulas: hexagon (2D) vs. octahedron (3D)
  - e.g. set of 8 formulas: octagon (2D) vs. cube (3D)

- logical requirement: diagrams that are closed under negation
  $\Rightarrow$ set of $2n$ formulas = set of n pairs of contradictory formulas (PCDs)
- geometrical requirement: PCDs share a central symmetry point
- nearly all Arist. diagrams in the literature satisfy these requirements
- example: Sesmat-Blanché hexagon for modal logic S5



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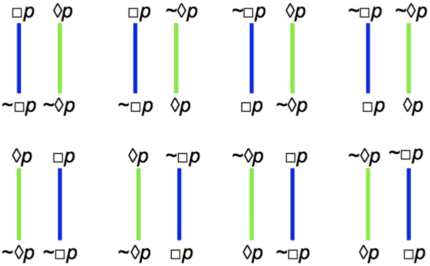
- number of configurations of n PCDs: $n! \cdot 2^n$
  - $n!$ permutations of the n PCDs
  - each of the n PCDs can be put into the configuration in 2 ways: $(\varphi, \neg\varphi)$ or $(\neg\varphi, \varphi)$ $\quad \Rightarrow 2 \times 2 \times \cdots \times 2 = 2^n$
- independent of concrete geometric visualization
- some concrete examples:

$n = 2$: $\quad 2! \cdot 2^2 = 2 \cdot 4 = 8$

$n = 3$: $\quad 3! \cdot 2^3 = 6 \cdot 8 = 48$

$n = 4$: $\quad 4! \cdot 2^4 = 24 \cdot 16 = 384$



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Geometrical symmetries

- every diagram has a number of symmetries
  (= cardinality of its symmetry group)
- visualize n PCDs using a diagram D that has k symmetries
  $\Rightarrow \frac{n! \cdot 2^n}{k}$ fundamentally distinct presentations of D
- two presentations are fundamentally distinct iff
  one cannot be obtained by reflecting and/or rotating the other
- general aim:

  fundamental **geometrical** differences should correspond
  (as much as possible) with **logical** differences

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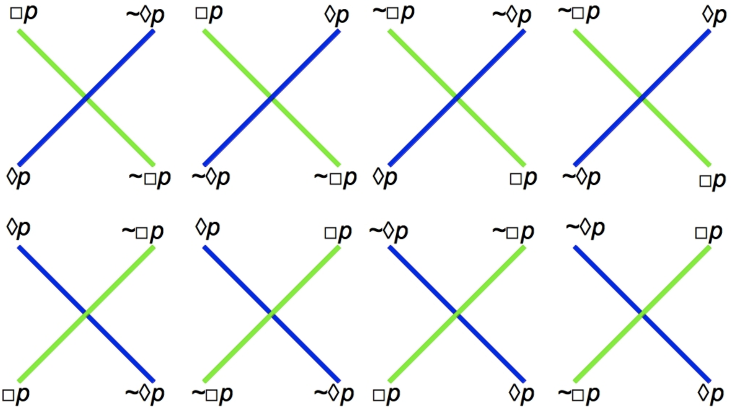
- given is a fixed set of 4 formulas, i.e. 2 PCDs
  $\Rightarrow 2! \cdot 2^2 = 8$ abstract configurations
- visualize these 8 abstract configurations using ordinary squares



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- these 8 squares are symmetric and/or rotational variants of each other
- this should not be surprising:
  - symmetry group of the square: dihedral group D_4
  - D_4 has 8 elements
- $\frac{2!\cdot 2^2}{|D_4|} = \frac{8}{8} = 1$ fundamental presentation
- there is exactly one way (up to symmetries and rotations) of visualizing 4 formulas using a square

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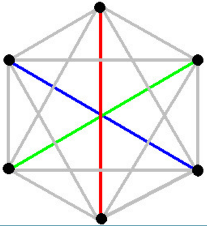
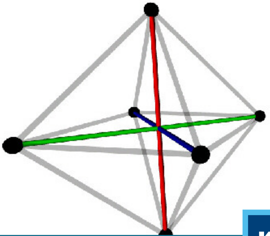
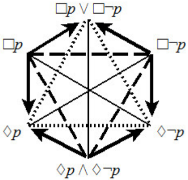
- fixed set of 6 formulas/3 PCDs $\Rightarrow 3! \cdot 2^3 = 48$ abstract configurations
- visualize these 48 abstract configurations using 48 **hexagons**
  - symmetry group of the hexagon: D_6: 12 symmetries
    $\Rightarrow \frac{48}{12} = 4$ fundamental presentations of the hexagon
- visualize these 48 abstract configurations using 48 **octahedrons**
  - symmetry group of the octahedron: O_h: 48 symmetries
    $\Rightarrow \frac{48}{48} = 1$ fundamental presentation of the octahedron

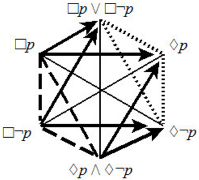


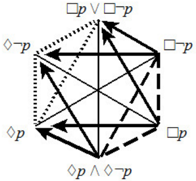


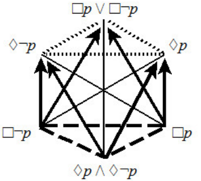
Case study 1: Jacoby-Sesmat-Blanché σ_3

- 6 formulas: $\Box p, \Box\neg p, \Diamond p, \Diamond\neg p, \Box p \vee \Box\neg p, \Diamond p \wedge \Diamond\neg p$
- 4 fundamental presentations of the hexagon:









	1	2	3	4
SA direction	chaos	uniform	uniform	uniform
C/SC triangle	equilateral	isosceles	isosceles	isosceles
center	exclusively C/SC	mainly SA	mainly SA	mainly SA
periphery	exclusively SA	mainly C/SC	mainly C/SC	mainly C/SC
long	exclusively C/SC	mainly SA	mainly SA	mainly SA
short	exclusively SA	mainly C/SC	mainly C/SC	mainly C/SC

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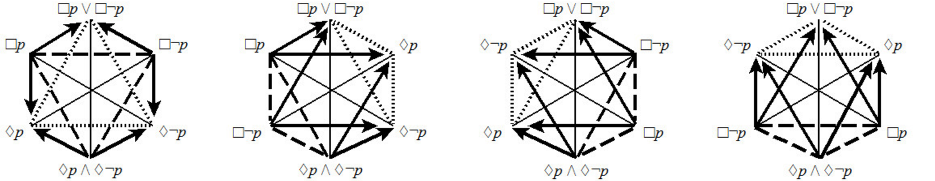

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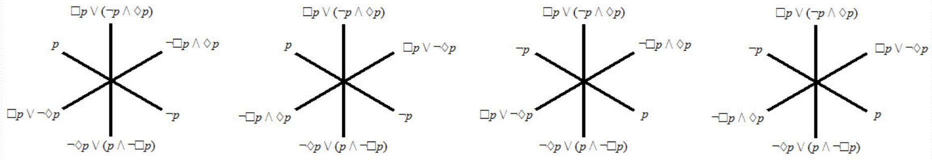
long †	exclusively C/SC	mainly SA	mainly SA	mainly SA
short ††	exclusively SA	mainly C/SC	mainly C/SC	mainly C/SC

Case study 2: fully degenerated σ_3

- 6 formulas:
  $p, \neg p, \Box p \vee \neg\Diamond p, \neg\Box p \wedge \Diamond p, \Box p \vee (\neg p \wedge \Diamond p), \neg\Diamond p \vee (p \wedge \neg\Box p)$
- fully degenerated:
  - $\frac{6 \cdot 5}{2} = \frac{30}{2} = 15$ pairs of formulas
  - 3 pairs: contradictory ($\Rightarrow$ 3 PCDs $\Rightarrow \sigma_3$)
  - 12 other pairs: no Aristotelian relation whatsoever (unconnectedness)
- 4 fundamental presentations of the hexagon:



- no differences between the 4 presentations whatsoever!

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- Jacoby-Sesmat-Blanché σ_3
  - 4 fundamental presentations of the hexagon: geometrical differences
  - corresponding logical differences: 1 vs 2,3,4

  $\Rightarrow$ hexagon visualization is preferred!

- fully degenerated σ_3 (12 $\times$ unconnectedness)
  - 4 fundamental presentations of the hexagon: geometrical differences
  - no corresponding logical differences whatsoever

  $\Rightarrow$ octahedron visualization is preferred!

- what about other types of σ_3?
  - Sherwood-Czezowksi
  - minimally degenerated (4 $\times$ unconnectedness)
  - intermediately degenerated (8 $\times$ unconnectedness)

Structure of the talk

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- systematic study of the different diagrams for a fixed set of formulas

- general case: $\sigma_n \Rightarrow$ concrete visualizations vs abstract mathematics
- a simple two-dimensional regular $2n$-gon has $4n$ symmetries
  $\Rightarrow \frac{n!\cdot 2^n}{4n} = (n-1)! \cdot 2^{n-2}$ fundamental presentations
- in abstract n-dimensional space: cross-polytope
  - dual of the n-dimensional hypercube
  - centrally symmetric polytope with $2n$ vertices and $n! \cdot 2^n$ symmetries

  $\Rightarrow \frac{n!\cdot 2^n}{n!\cdot 2^n} = 1$ fundamental presentation

- concrete illustration: σ_4 (Buridan, Béziau, Moretti, ...)
  - ▷ 2D octagon $\frac{4!\cdot 2^4}{4\cdot 4} = \frac{384}{16} = 24$ fundamental presentations
  - ▷ 4D 16-cell $\frac{4!\cdot 2^4}{4!\cdot 2^4} = \frac{384}{384} = 1$ fundamental presentation

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  - ▷ 2D octagon $\frac{4!\cdot 2^4}{4\cdot 4} = \frac{384}{16} = 24$ fundamental presentations
  - ▷ 3D cube $\frac{4!\cdot 2^4}{48} = \frac{384}{48} = 8$ fundamental presentations
  - ▷ 4D 16-cell $\frac{4!\cdot 2^4}{4!\cdot 2^4} = \frac{384}{384} = 1$ fundamental presentation

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Thank you!

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