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The Unreasonable Effectiveness of Bitstrings in Logical Geometry

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The central aim of Logical Geometry is

- to develop an *interdisciplinary framework*
- for the study of *geometrical representations*
- in the analysis of *logical relations*.

More in particular:

- we analyse the **logical relations** of opposition, implication and duality between expressions in various logical, linguistic and conceptual systems.
- we study abstract **geometrical representations** of these relations as well as their visualisation by means of 2D and 3D diagrams.
- we develop an **interdisciplinary framework** integrating insights from logic, formal semantics, algebra, group theory, lattice theory, computer graphics, cognitive psychology, information visualisation and diagrams design.

Bitstrings

- are an extremely powerful tool
- yield both quantitative and qualitative results
- raise interesting new questions

Main aims of the talk:

- provide a unified account of bitstrings in logical geometry
- illustrate their effectiveness on different levels

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Bitstrings

Bitstrings are sequences of bits (0/1) that encode (denotations of) formulas

Modal Logic S5	Propositional Logic	bitstrings level 1	bitstrings level 3	Propositional Logic	Modal Logic S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$\neg\Box p \wedge p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\Box p \vee \neg p$
$\Diamond p \wedge \neg p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$\neg\Diamond p \vee p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$

Modal Logic S5	Propositional Logic	bitstrings level 2/0	bitstrings level 2/4	Propositional Logic	Modal Logic S5
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\Diamond p \wedge \neg p)$	q	1010	0101	$\neg q$	$\neg\Diamond p \vee (\neg\Box p \wedge p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

Bitstrings have been used to encode

- **logical systems**: e.g. classical propositional logic, first-order logic, modal logic and public announcement logic
- **lexical fields**: e.g. comparative quantification, subjective quantification, color terms and set inclusion relations

Remark:

- we use bitstrings to encode **formulas**, not **relations** between formulas
- if a formula φ is encoded by the bitstring b, we write $\beta(\varphi) = b$

Relative to a Boolean logical system S, two **formulas** φ, ψ are

contradictory	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$

In terms of bitstrings, two **bitstrings** b_1 and b_2 are

contradictory	iff	$b_1 \wedge b_2 = 0000$	and	$b_1 \vee b_2 = 1111$
contrary	iff	$b_1 \wedge b_2 = 0000$	and	$b_1 \vee b_2 \neq 1111$
subcontrary	iff	$b_1 \wedge b_2 \neq 0000$	and	$b_1 \vee b_2 = 1111$
in subalternation	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 \neq b_1$

- φ and ψ stand in some Aristotelian relation (defined for S) iff $\beta(\varphi)$ and $\beta(\psi)$ stand in that same relation (defined for bitstrings).
- β maps formulas from S to bitstrings, preserving Aristotelian structure (Representation Theorem for finite Boolean algebras)

- In most cases, the mapping β assigns a *semantics* to the formulas (>< Pellissier's setting approach).
- Each bit provides an answer to a (binary) meaningful question (analysis of generalized quantifiers as sets of sets).
- In S5 the bit positions encode answers to the following questions:

Is φ true if

p is true in all possible worlds?	yes/no
p is true in the actual world but not in all possible worlds?	yes/no
p is true in some possible worlds but not in the actual world?	yes/no
p is true in no possible worlds?	yes/no

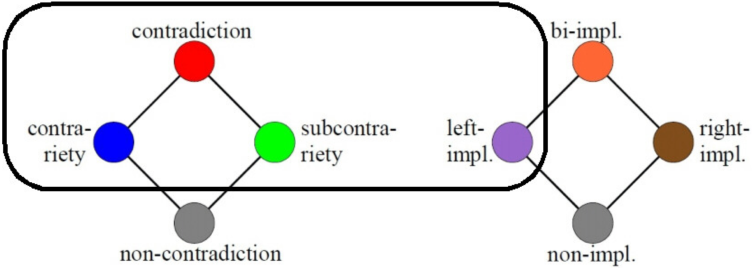
- Examples:

$$\beta(\Diamond p) = 1110 = \langle \text{ yes, yes, yes, no } \rangle$$

$$\beta(\Diamond p \wedge \Diamond \neg p) = 0110 = \langle \text{ no, yes, yes, no } \rangle$$

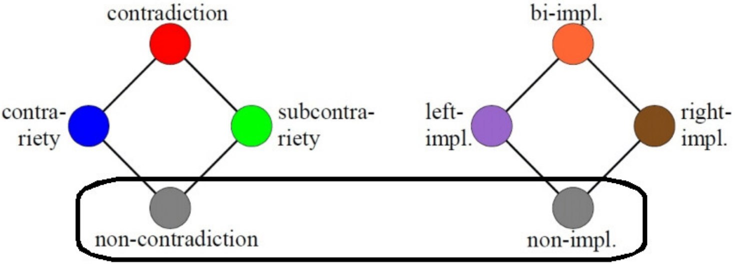
$$\beta(\Diamond \neg p) = 0111 = \langle \text{ no, yes, yes, yes } \rangle$$

- the set of 4 **Aristotelian** relations is hybrid between two other sets of logical relations that are ordered by information level
- **Opposition** relations:
  contradiction, contrariety, subcontrariety, and non-contradiction
- **Implication** relations:
  bi-implication, left-implication, right-implication, and non-implication



Unconnectedness (logical independence):

- absence of any Aristotelian relation
- combination of least informative Opposition and Implication relations



- Unification: Unconnectedness requires bitstrings of length at least 4
- Theorem: φ and ψ unconnected $\Rightarrow$ $\beta(\varphi)$ and $\beta(\psi)$ have $\geq$ 4 bits

For any bitstring of length n and level i we can use simple combinatorial arguments to calculate the number of

contradictories	$\#CD$	$= 1$
contraries	$\#C$	$= 2^{n-i} - 1$
subcontraries	$\#SC$	$= 2^{i} - 1$
non-contradictories	$\#NCD$	$= (2^{n-i} - 1)(2^{i} - 1)$

- Note that $\#CD < \#C, \#SC < \#NCD$ iff $1 < i < n - 1$
- Recall informativity ordering: $CD > C, SC > NCD$

- Note that if $i \approx \frac{n}{2}$, then $\#C \approx \#SC$
- Bitstrings in middle levels have similar numbers of contraries and subcontraries; recall informativity ordering: $C \equiv SC$

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- Boolean closure of bitstrings length 4 $\overset{2009}{\Longrightarrow}$ rhombic dodecahedron (RDH)
- internal structure of RDH $\overset{2013}{\Longrightarrow}$
  - exhaustive typology of Aristotelian diagrams for length 4 bitstrings
  - CO perspective: cube (L1-L3) + octahedron (L2-L2)

- Use bitstrings to study embeddings
  - rhombic dodecahedron $\sim$ bitstrings of length 4
strong JSB hexagon $\sim$ bitstrings of length 3

  - compression of bitstrings: length 4 $\rightsquigarrow$ length 3
  - e.g. $b_1 = b_2$: **11**00 $\rightsquigarrow$ **1**00, **00**10 $\rightsquigarrow$ **0**10, **00**11 $\rightsquigarrow$ **0**11

  - 6 strong JSB hexagons in RDH $\sim$ 6 compressions length 4 $\rightsquigarrow$ length 3

  - $b_2 = b_3$, (1950s) $b_1 = b_2$, (2003) $b_3 = b_4$, (2003) $b_1 = b_4$, (2005*) $b_1 = b_3$, (2005) $b_2 = b_4$ (2005)

How many hexagons can be constructed with bitstrings of length ℓ?

- 2^ℓ bitstrings of length $\ell \rightsquigarrow (2^\ell - 2)$ contingent bitstrings of length ℓ
- bitstrings are chosen in contradictory pairs: $\frac{(2^\ell-2)(2^\ell-4)(2^\ell-6)}{48}$
-

$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$	$\ell = 7$
$\frac{(\mathbf{6})(4)(2)}{48}$	$\frac{(\mathbf{14})(12)(10)}{48}$	$\frac{(\mathbf{30})(28)(26)}{48}$	$\frac{(\mathbf{62})(60)(58)}{48}$	$\frac{(\mathbf{126})(124)(122)}{48}$
1	35	455	4495	39711

- computational importance of bitstrings for generating hexagons.

Different types of hexagons require bitstrings of different length:

- strong Jacoby-Sesmat-Blanché (JSB) requires length 3
- weak JSB, Sherwood-Czezowski, U4 and U12 require length 4
- U8 requires length 5
- no hexagons require length 6, 7 ...

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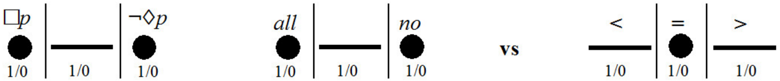
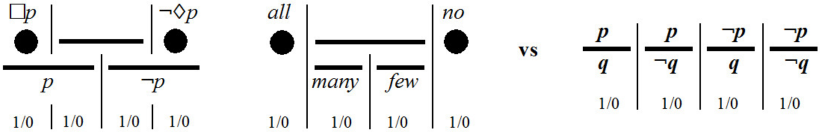
- Bitstrings generate new questions about
  - the linguistic/cognitive aspects of the expressions they encode
  - the relative weight/strength of individual bit positions inside bitstrings
  - the underlying scalar/linear structure of the conceptual domain

- Edges versus center in bitstrings of length 3



- Bitstrings of length 4 as refinements/expansions of bitstrings of length 3



- From mathematical/algebraic perspective no difference (so far) between
  - 'linear' bitstrings (such as 1010)
  - 'non-linear' bitstrings (such as $1^0_1 0$)
- From linguistic/cognitive perspective difference is relevant :
  - Linear bitstrings imply that all questions (all bits) about a lexical field can be situated on a single dimension
    $\rightsquigarrow$ comparative quantification, proportional quantification, propositional connectives, *all/many*$_2$*/few*$_2$*/no*
  - Non-linear bitstrings imply that the various questions belong to fundamentally distinct dimensions
    $\rightsquigarrow$ modality in $S5$, *all/John/not-John/no*, *all/many*$_1$*/few*$_1$*/no*
  - Formulate empirical hypotheses concerning the cognitive complexity (e.g. processing times) of these lexical fields.
    $\rightsquigarrow$ future research

- Aristotelian relations between bitstrings in Logical Geometry
- Logical Effectiveness
  - Unconnectedness (non-contradiction + non-implication) length ≥ 4
  - Counting (sub)contraries: $\#CD < \#C, \#SC < \#NCD$
- Diagrammatic Effectiveness
  - 6 strong JSB hexagons in RDH $\sim$ 6 compressions length 4 $\rightsquigarrow$ length 3
  -

length 3	length 4	length 5
strong JSB	weak JSB, Sherwood-Czezowski Unconnected4, Unconnected12	Unconnected8

- Linguistic/Cognitive Effectiveness
  - scales length 4 as refinement of length 3
  - 'linear bitstrings $\sim$ 1 dimension' vs 'non-linear bitstrings $\sim \neq$ dimensions'

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Thank you!

More info: www.logicalgeometry.org